Certain modulus estimate on arbitrary Riemann surfaces(Function Spaces on Riemann Surfaces)

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Certain modulus estimate on arbitrary Riemann surfaces

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§ 1. Definitions and main Theorem.

Let R be an arbitrary Riemann surface and c be a simple closed curve on R. Set

$$D(c,R) = \{\rho: \rho = \rho(z)|dz| \text{ is a measurable (conformal pseudo-) metric on } R \text{ with } \rho(z) \geq 0 \text{ such that } \int_{c'} \rho \geq 1 \text{ for every closed curve } c' \text{ freely homotopic to } c \text{ on } R\},$$

and call the quantity

$$M(c,R) = \inf_{\rho \in D(c,R)} A(\rho)$$

the modulus of the free homotopy class of c on R, where $A(\rho) = \iint_R \rho(z)^2 dxdy$ is the area of R with respect to the metric $\rho(z)|dz|$. Recall that the quantity $1/M(c,R)$ is the extremal length of the same class.

Similarly, we set

$$D'(c,R) = \{\rho: \rho = \rho(z)|dz| \text{ is a measurable}$$

metric on R with $\rho(z) \geq 0$ such that $\int_{c'} \rho \geq 1$ for every 1-cycle c' homologous to c on $R \}$,

and call the quantity

$$M'(c,R) = \inf_{\rho \in D'(c,R)} A(\rho)$$

the modulus of the homology class of c on R.

Here, if $D(c,R)$ and/or $D'(c,R)$ are/is empty, then we assume that $M(c,R)$ and/or $M'(c,R)$ are/is equal to $+\infty$.

Now it is clear that

$$M'(c,R) \geq M(c,R) ,$$

for $D'(,cR)$ is contained in $D(c,R)$. But in general, we can not bound $M(c,R)$ from below by any quantity depending only on $M'(c,R)$, which can be seen by an easy example. So, it seems interesting to find another quantity which is related closely to the homology class of c on R, but behaves like $M(c,R)$.

For this purpose, first we recall that $M(c,R)$ and $M'(c,R)$ can be represented as norms of certain (extremal) differentials on R, when they are finite.

The differential for $M(c,R)$ is the so-called Jenkins-Strebel's differential, i.e. a holomorphic quadratic differential $\phi = \phi(c,R)$ on R with closed trajectories such that $\rho_c = |\phi|^{1/2}$ belongs to $D(c,R)$ and $M(c,R) = A(\rho_c)$. And, from geometrical viewpoint, $M(c,R)$ is the supremum of the (geometrical) modulus of all ring domains "homotopic" to c on R, and the above ϕ gives the extremal ring domain for c on R (, cf. [2] and [4]).

On the other hand, the differential for $M'(c,R)$ is the period reproducer for c. Namely, let $\sigma = \sigma(c,R)$ be a square integrable real harmonic differential on R which is uniquely determined by the condition that

$$\int_c \omega = (\omega, \sigma)_R \quad (= \iint_R \omega \wedge {}^*\sigma)$$

for every real square integrable harmonic differential ω on R. Then Accola's theorem ([1]) states that the extremal length $1/M'(c,R)$ of the homology class of c on R is equal to $\|\sigma\|_R^2 = (\sigma, \sigma)_R$, or more precisely, that $\rho'_c = |\sigma + i^*\sigma|/\|\sigma\|_R^2$ belongs to $D'(c,R)$ and $M'(c,R) = A(\rho'_c)$.

Here, $\sigma \equiv 0$ if and only if $M'(c,R) = +\infty$. In the sequel, we assume that $\sigma(c,R) \not\equiv 0$ (, i.e. $M'(c,R) < +\infty$), and call c homologically non-degenerate. Then it is known ([7, Proposition 2]) that $\theta(c,R) = \sigma(c,R) + i^*\sigma(c,R)$ has closed trajectries.

Here a compact regular trajectory of a holomorphic abelian differential θ ($\not\equiv 0$) is a simple closed curve γ such that θ has no critical point on γ and $\operatorname{Im}\theta \equiv 0$ along γ. And we say that θ has closed trajectories if the complement of all compact regular trajectories of θ has area 0.

And it is natural to consider the ring domain $W(c,R)$ on R swept out by all compact regular trajectories of $\theta(c,R)$ freely homotopic to c (with suitable orientation). Let $M''(c,R)$ be the modulus of $W(c,R)$ (, where we set $M''(c,R) = 0$ when $W(c,R)$ is empty), then $M''(c,R)$ relates closely to the free homotopy class of c , while $\Theta(c,R)$ itself depends only on the homology class of c . Actually we can show the following

Theorem 1 ([8]). There is an absolute constant A such that for every Riemann surface R and every homologically non-degenerate c on R , it holds that

$$M''(c,R) \leq M(c,R) \leq M''(c,R) + A .$$

Remark. Even in homologically degenerate (but homotopically non-degenerate) case, we can show a similar result as Theorem 1, by considering suitable meromorphic differentials of the third kind instead of $\Theta(c,R)$. See [8, § 3].

An outline of proof of Theorem 1 will be given in § 2.

And an application of Theorem 1 will be considered in § 3, where we will give a characterization of convergent sequences in the finitely augmented Teichmüller spaces (, i.e. a characterization of the conformal topology. See Theorem 2 and Remark at the end of § 3).

Finally, in § 4, we also include some remarks concerning on continuity of differentials.

§ 2. Proof of Theorem 1.

To prove Theorem 1, we first recall certain metrical property of period reproducers.

Fix R and c as in Theorem 1, then it is well-known that $*\sigma(c,R)$ has integral periods, i.e. the period of $*\sigma(c,R)$ along any 1-cycle is an integer . Hence we can consider a mapping

$$u(p) = \exp(\ 2\pi i \cdot \int^{p} *\sigma(c,R)\ )$$

from R into the unit circle $S^1 = \{|z| = 1\}$ (, which is, in [5], called a circular function for $*\sigma(c,R)$), and we know the following

Proposition 1 ([5]). For every $t \in S^1$, let L_t be the set of all (not necessarily compact) regular trajectories of $\theta(c,R)$ contained in $u^{-1}(t)$ (, which is called the set of level curves of $\theta(c,R)$ for t), and $m(t)$ be the total length of L_t with respect to the metric $|\theta(c,R)|$, i.e.

$$m(t) = \int_{L_t} |\theta(c,R)| \ (= \int_{L_t} |\sigma(c,R)| \ ).$$

Then ($m(t) \leq \|\sigma(c,R)\|_R^2$ for every $t \in S^1$, and) it holds that

$$m(t) = \|\sigma(c,R)\|_R^2 \quad (= \int_c \sigma(c,R) \ )$$

for every $t \in S^1$ except for at most one value.

Corollary. Let $t \in S^1$ be fixed. If L_t contains a compact regular trajectory, say c_t , of $\theta(c,R)$ freely homotopic to c , then it holds that $L_t = \{c_t\}$.

PROOF. The assertion follows from the above Proposition 1 by noting that

$$m(t) \geq \int_{c_t} |\sigma(c,R)| \geq \left| \int_c \sigma(c,R) \right| \geq \|\sigma(c,R)\|_R^2$$

under the assumption in Corollary. q.e.d.

Here we recall the following

Definition ([8]). Let $V_r = \{1<|z|<r\}$ for arbitrarily given r (>1). We call a harmonic function $u(z)$ in a neighbourhood of $\overline{V_r}$ 'a height function' on V_r if $u(z)$ satisfies the following conditions;

(1) $\left| \int_c *du \right| = 1$ for every non-trivial dividing curve c in V_r,

(2) it holds that $\int_{L_t} |*du| \leq 1$ for every t, and

(3) if L_t conatins a simple closed curve, say c_t, then $L_t = \{c_t\}$, where L_t is defined as before (for the differential $*du$).

Then the following comparison theorem for height functions is known.

Proposition 2 ([8]). There is an absolute constant B such that for every V_r and every height function $u(z)$ on V_r, it holds that

$$M(u) \leq (1/2\pi)\cdot\log r \leq M(u) + B ,$$

where $M(u)$ is the length of the maximal interval I such that, for every $t \in I$, L_t consists of one simple closed curve.

PROOF OF THEOREM 1. The first inequality in Theorem 1 is clear from the definition of $M''(c,R)$ and Jenkins-Strebel's theorem.

To show the second inequality, fix a positive ε arbitrarily, then from the geometrical characterization of Jenkins-Strebel's differential $\phi(c,R)$, we can construct a ring domain W_ε such that any non-trivial simple closed curve on W_ε is freely homotopic to c or $-c$ on R, and that the modulus $M(W_\varepsilon)$ of W_ε is not less than $M(c,R) - \varepsilon$.

Next consider a harmonic function

$$u_c(p) = (1/\|\sigma(c,R)\|_R^2)\cdot\int_{p_0}^{p} {}^*\sigma(c,R)$$

on W_ε with a fixed $p_0 \in W_\varepsilon$. Then, mapping W_ε onto a suitable annulus, we see from Proposition 1 and Corollary that $u_c(p)$ can be considered as a height function on some V_r with $(1/2\pi)\cdot\log r \geq M(W_\varepsilon) - \varepsilon$. Hence by Proposition 2, we have that

$$M(u_c) + B \geq (1/2\pi)\cdot\log r \geq M(W_\varepsilon) - \varepsilon \geq M(c,R) - 2\varepsilon .$$

And from the definition of $u_c(p)$, it is easily seen that

$$M''(c,R) \geq M(u_c) .$$

Since ε is arbitrary, we have that

$$M''(c,R) + B \geq M(c,R) .$$

q.e.d.

§ 3. An application of Theorem 1.

Fix a Riemann surface R^* such that the (reduced) Teichmüller space $T(R^*)$ of R^* is non-trivial. And consider the finitely augmented Teichmüller space $\hat{T}(R^*)$ of R^*. (For the details, see [7, § 1.1°)].) Here we recall some of definitions.

First, $\hat{T}(R^*)$, as a point set, is the set of all marked Riemann surfaces R with at most a finite number of nodes that admit marking-preserving deformations $(f;R^*,R)$.

Here a deformation $(f;R_1,R_2)$ of R_1 to R_2, where R_1 and R_2 are Riemann surfaces with at most (a finite number of) nodes, is a continuous surjection from R_1 onto R_2 such that

(i) $f^{-1}(p)$ is a node of R_1 or a simple closed curve on R_1 for every node p of R_2, and

(ii) f^{-1} is quasiconformal on $R_2 - \bar{U}$ for every neighborhood U of $N(R_2)$, where and in the sequel, $N(R)$ is the set of all nodes of R.

Next, following Abikoff, $\hat{T}(R^*)$ is equipped with the conformal topology. Recall that a sequence $\{R_n\}_{n=1}^{\infty}$ in $\hat{T}(R^*)$ converges to $R_0 \in \hat{T}(R^*)$ if and only if there is an admissible sequence $\{(f_n;R_n,R_0)\}_{n=1}^{\infty}$ of marking-preserving deformations of R_n to R_0, i.e., such a sequence that, for every $\varepsilon > 0$ and every neighborhood U of $N(R_0)$, there is an N such that $(f_n)^{-1}$ is $(1+\varepsilon)$-quasiconformal on $R_0 - \bar{U}$ for every $n \geq N$.

Now we return to the modulus $M''(c,R)$, and fix a homologically non-degenetare curve c (on R^*, hence on every R in $T(R^*)$) arbitrarily. Then we know the following

Proposition 3. The modulus $M''(c,R)$ is continuous on $T(R^*)$.

PROOF. The assertion follows from geometrical continuity of period reproducers ([6, Theroem 5]). q.e.d.

And Theorem 1 gives the following

Proposition 4. Denote

$$\partial_c T(R^*) = \{R \in \hat{T}(R^*) : N(R) \text{ consists of a single node } p(R) \text{ corresponding to } c\},$$

and set $M''(c,R) = +\infty$ for every $R \in \partial_c T(R^*)$.

Suppose that $R_n \in \hat{T}(R^*)$ converges to $R_0 \in \partial_c T(R^*)$, then ($_c\hat{T}(R^*) = T(R^*) \cup \partial_c T(R^*)$ contains R_n for every sufficiently large n , and) it holds that

$$\lim_{n \to +\infty} M''(c,R_n) = +\infty .$$

PROOF. Since $M(c,R_n)$ clearly tends to $+\infty$ (, which can be seen from the definition of the conformal topology), the assertion follows from Theorem 1. q.e.d.

In particular, letting $\{R_n\}_{n=0}^{\infty}$ be as in Proposition 4, $W(c,R_n)$ is non-empty for every sufficiently large n . So we set

$$S_c = \{R \in T(R^*) : W(c,R) \text{ is non-empty, or equivalently, } M''(c,R) > 0\},$$

then there is a neighborhood V of R_0 in $\hat{T}(R^*)$ such that $V \cap T(R^*) = S_c$.

Moreover, for every $R \in S_c$, there is a natural surgery # from S_c onto $\partial_c T(R^*)$ defined by the following three steps.

1) Cut R along the center trajectory, say C_R , of $W(c,R)$.

2) Paste two copies of once punctured disks along the borders of $R - C_R$ corresponding to C_R so that $\theta(c,R)$ restricted on $R - C_R$ can be extended to a holomorphic differential on the whole resulting surface, say R' .

3) Fill two punctures of R' corresponding to those of pasted disks with one point $p(R)$.

The resulting surface $R^{\#}$, with the natural marking, can be considered as a point of $\partial_c T(R^*)$ (with the node $p(R)$ corresponding to c), and set $\#(R) = R^{\#}$.

Using this operation # , we can show the following

Theorem 2 ([9]). $R_n \in T(R^*)$ converges to $R_0 \in \partial_c T(R^*)$ in the sense of the conformal topology if and only if

i) $\lim_{n \to +\infty} M''(c,R_n) = +\infty$, and

ii) $\#(R_n)$ converges to R_0 in $\partial_c T(R^*)$ (, or equivalently, $\#(R_n) - N(\#(R_n))$ converges to $R_0 - N(R_0)$ in the sense of the usual Teichmüller topology).

PROOF. Suppose that R_n converges to R_0 in $\hat{T}(R^*)$, then the assertion i) follows from Proposition 4. The assertion ii) can be shown by using Proposition 2 and [6, Theorem 1] (, but we omit the details).

Conversely, suppose that i) and ii) holds. Then by using [6, Propositions 5 and 6 and Lemma 7], we cab easily construct an admissible sequence $\{(f_n;R_n,R_0)\}_{n=1}^{\infty}$. q.e.d.

The case of homologically degenerate (, but homotopically non--degenerate) curve c can be treated similarly. For example, suppose that R^* admits Green's functions. (The case that R^* admits no Green's functions is parallel.) Then c is dividing, and one of components, say S , of $R^* - c$ is a subregion of type SO_{HB} (, i.e. a parabolic end) which is not simply connected. In this case, distinguishing one point (on $S \subset R^*$, hence on every $R \in {}_c\hat{T}(R^*)$), and writting by g_R Green's function on R ($\in T(R^*)$) with the pole at the distinguished point, we can consider the characteristic ring domain $W(c,R)$ of the differential $-{}^*dg_R + idg_R$ (instead of $\Theta(c,R)$) for c on R for every $R \in T(R^*)$ (, cf. [5, Corollary 1]). Using this $W(c,R)$, we define $M''(c,R)$ and the operation $\#$, and conclude a similar characterization as Theorem 2 of convergence sequences in ${}_c\hat{T}(R^*)$.

Finally, we note here that, in any case (of a homotopically non-degenerate curve), the operation $\#$ is just the inverse of Schiffer-Spencer's variation by reopening a node (, which is investigated in [3] in case of finite Riemann surfaces).

Remark. Generalizing above investigation, we can give a characterization of the conformal topology on the whole finitely augmented Teichmüller space (as the topology induced from variation by reopening a finite number of nodes, which will appear in [10]).

§ 4. Closing remarks (on continuity of differentials).

1) As another application of Theorem 1, we can show certain locally uniform boundedness of Green's functions on the finitely augmented Teichmüller space. (Here, for every $R \in \hat{T}(R^*)$ and every $q \in R - N(R)$, Green's function $g(p,q;R)$ on R with the pole q is, by definition, equal to the usual one or identically zero on the component S of $R - N(R)$ containing q, according as S admits Green's functions or not, and is identically zero on $R-S$.) And using this, we can show the following continuity of Green's functions.

<u>Proposition 5 ([9]).</u> Let an admissible sequence $\{(f_n;R_n,R_0)\}_{n=1}^{\infty}$ of marking-preserving deformations and a point $q \in R_0 - N(R_0)$ be given. Suppose that $g(p,q;R_0) \not\equiv 0$ and that there is a neighborhood U_q of q on $R_0 - N(R_0)$ and an n_0 such that $(f_n)^{-1}$ is conformal on U_q for every $n \geq n_0$. Then

$$\varphi_n = dg(\cdot,(f_n)^{-1}(q);R_n) + i{*}dg(\cdot,(f_n)^{-1}(q);R_n)$$

converges to

$$\varphi_0 = dg(\cdot,q;R_0) + i{*}dg(\cdot,q;R_0)$$

strongly metrically with respect to $\{f_n\}$, i.e. it holds that

$$\lim_{n\to+\infty} \|\varphi_n \circ (f_n)^{-1} - \varphi_0\|_{R_0 - U} = 0$$

for every neighborhood U of $N(R_0)$, where $\varphi \circ f$ is the pull--back of φ by f.

Remark. Several results on (strongly) metrical convergence of differentials (including period reproducers) have been obtained in [7, § 3].

2) The essential part (, i.e. the "only if" part) of Theorem 2 can be restated, again, as a theorem on metrical continuity of certain differentials. To state it precisely, set

$$\theta(R) = \theta(c,R)/\|\sigma(c,R)\|_R^2$$

for every $R \in T(R^*)$. And letting $p_1(R)$ and $p_2(R)$ be punctures of $R' = R - N(R)$ corresponding to c for every $R \in \partial_c T(R^*)$, we set

$$\begin{aligned}\theta(R) &= \phi_{p_1(R),R'} - \phi_{p_2(R),R'} \text{ , or} \\ &= \phi_{p_1(R),p_2(R),R'}\end{aligned}$$

according as R admits Green's functions or not, where $\phi_{p,R}$ and $\phi_{p_1,p_2,R}$ are defined in [6, § 3], and $p_1(R)$ and $p_2(R)$ are so chosen that $\int_c \theta(R) = 1$.

Then we have the following version of the "only if" part of Theorem 2.

Proposition 6 ([9]). Suppose that R_n converges to R_0 in ${}_c\hat{T}(R^*)$, then $\theta(R_n)$ converges to $\theta(R_0)$ strongly metrically (with respect to any admissible sequence $\{(f_n;R_n,R_0)\}_{n=1}^{\infty}$).

3) One may consider that (strongly) metrical continuity of prescribed differentials is a much weaker result than any variation formula for them. But in several typical cases, if the differentials satisfy a certain orthogonality condition (and natural boundedness conditions, which sometimes hold ovbiously), we can obtain certain kinds of first variation formulas for them.

Some of variation formulas under quasiconformal deformation of Riemann surfaces can be derived on this line, and have been investigated by Y. Kusunoki, F. Maitani, the author, and others (, cf. References of [7]).

The proofs of variation formulas given in [9] (for period reproducers and Green's functions under classical Schiffer--Spencer's variation by reopening a node) can be considered as another case in which the above principle is available.

Finally, the author announces that more general variation formulas under a certain kind of pinching deformation (which can be shown on the same lines as above) will be investigated in [10].

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